The Geometric Means in Banach $*$-Algebras

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Dedicated to Professor Joe Diestel for his 60^{th} birthday

Abstract. The arithmetic-geometric-harmonic inequality has played a special role in elementary mathematics. During the past twenty five years (see [1], [2] and [8] etc.) a great many mathematicians have researched on various kinds of matrix versions of the arithmetic-geometric-harmonic inequality. In this article we will define the geometric means of positive elements in Banach $*$-algebras and prove that the arithmetic-geometric-harmonic inequality does hold in Banach $*$-algebras.

Keywords and Phrases. Arithmetic mean, geometric mean, harmonic mean and Banach $*$-algebra.

2000 AMS Subject Classification. 47A63, 47A64.

1. Introduction

Let A be a Banach $*$-algebra. An element $a \in A$ is called *self-adjoint* if $a^* = a$. A is *Hermitian* if every self-adjoint element a of A has real spectrum: $\sigma(a) \subset \mathbb{R}$, where $\sigma(a)$ denotes the spectrum of a. We assume in what follows that a Banach $*$-algebra A is Hermitian. Also we assume that A is unital with unit 1. Saying an element $a \geq 0$ means that $a = a^*$ and $\sigma(a) \subset [0, \infty)$. $a > 0$ means that $a \geq 0$ and $0 \notin \sigma(a)$. Thus, $a > 0$ implies its inverse a^{-1} exists. Denote the set of all invertible elements in A by $Inv(A)$. If $a, b \in A$, then $a, b \in \text{Inv}(A)$ imply $ab \in \text{Inv}(A)$, and $(ab)^{-1} = b^{-1}a^{-1}$. Saying $a \geq b$ means $a - b \geq 0$, and $a > b$ means $a - b > 0$. Shirali-Ford Theorem ([6] or [3], Theorem 41.5) asserted that $a^*a \geq 0$ for every $a \in A$. Based on Shirali-Ford Theorem, Okayasu [5], Tanahashi and Uchiyama [7] proved the following inequalities:

(1) If $a, b \in A$, then $a \geq 0,\ b \geq 0$ imply $a + b \geq 0$, with $\alpha \geq 0$ implies $\alpha a \geq 0$.
(2) If $a, b \in A$, then $a > 0,\ b \geq 0$ imply $a + b > 0$.
(3) If $a, b \in A$, then either $a \geq b > 0$, or $a > b \geq 0$ imply $a > 0$.
(4) If $a > 0$, then $a^{-1} > 0$.
(5) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$; Also $0 < b \leq a$ if and only if $cbc \leq cac$.
(6) If $0 < a < 1$, then $1 < a^{-1}$.
(7) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$; Also if $0 < b \leq a$, then $0 < a^{-1} \leq b^{-1}$.

Also, Okayasu [5] showed that the following Löwner-Heinz inequality still holds in Banach $*$-algebras:

Theorem Let A be a unital Hermitian Banach $*$-algebra with continuous involution. Let $a, b \in A$ and $p \in [0,1]$. Then $a^p > b^p$ if $a > b$, and $a^p \geq b^p$ if $a \geq b$.

It is natural to ask if there is an arithmetic-geometric-harmonic means inequality in Banach $*$-algebras?

In this paper, we will address this problem.

2. The laws of exponents

Let $a \in A$ and $a > 0$, then $0 \notin \sigma(a)$ and the fact of $\sigma(a)$ being nonempty compact subset of $\mathbb{C}$ implies that

$$\inf\{z : z \in \sigma(a)\} > 0 \qquad \text{and} \qquad \sup\{z : z \in \sigma(a)\} < \infty.$$

Choose γ to be a closed rectifiable curve in $\{\text{Re } z > 0\}$, the right half open plane of the complex plane, such that $\sigma(a) \subset \text{ins } \gamma$, the inside of γ. Let G be an open subset of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a) = \frac{1}{2\pi i} \int_\gamma f(z)(z-a)^{-1} dz.$$

It is known (see [4], p.201 - p.204) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem:

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

For any $\alpha \in \mathbb{R}$, we define

$$a^\alpha = \frac{1}{2\pi i} \int_\gamma z^\alpha (z-a)^{-1} dz$$

where z^α is the principal α-power of z. Since A is a Banach $*$-algebra, $a^\alpha \in A$. Since z^α is analytic in $\{\text{Re } z > 0\}$, by the Spectral Mapping Theorem

$$\sigma(a^\alpha) = (\sigma(a))^\alpha = \{z^\alpha : z \in \sigma(a)\} \subset (0, \infty).$$

Thus, we have

Lemma 1 *If* $0 < a \in A$ *and* $\alpha \in \mathbb{R}$, *then* $a^\alpha \in A$ *with* $a^\alpha > 0$.

Moreover, one of the laws of exponents holds in Banach $*$-algebras.

Lemma 2 *If* $0 < a \in A$ *and* $\alpha, \beta \in \mathbb{R}$, *then* $a^\alpha a^\beta = a^{\alpha+\beta}$.

Proof Let γ be defined as in the discussion preceding lemma 1. It is known that ([4], VII. 4.7. Riesz Functional Calculus) that the map

$$f \mapsto f(a) = \frac{1}{2\pi i}\int_\gamma f(z)(z-a)^{-1}dz$$

of $\mathrm{Hol}(a) \to A$ is an algebra homomorphism, where $\mathrm{Hol}(a)$ = all of the functions that are analytic in a neighborhood of $\sigma(a)$. That is, $f(a)g(a) = (fg)(a)$. Moreover, $z^\alpha z^\beta = z^{\alpha+\beta}$ holds for principal powers of z implies that

$$a^\alpha a^\beta = \frac{1}{2\pi i}\int_\gamma z^\alpha z^\beta (z-a)^{-1}dz = \frac{1}{2\pi i}\int_\gamma z^{\alpha+\beta}(z-a)^{-1}dz = a^{\alpha+\beta}.$$

□

Lemma 3 *If* $0 < a \in A$ *and* $\alpha \in \mathbb{R}$, *then* $(a^\alpha)^{-1} = (a^{-1})^\alpha = a^{-\alpha}$.

Proof Note that $a^0 = 1$ ([3], Lemma 1, p.31), and from Lemma 2 we have

$$a^\alpha a^{-\alpha} = a^{\alpha+(-\alpha)} = a^0 = 1.$$

By the uniqueness of the inverse of an element in A, $(a^\alpha)^{-1} = a^{-\alpha}$.

Next we want to verify that $(a^{-1})^\alpha = a^{-\alpha}$. We know that $a > 0$ implies that

$$\inf\{z : z \in \sigma(a)\} > 0 \qquad \text{and} \qquad \sup\{z : z \in \sigma(a)\} < \infty.$$

Choose positive real numbers r_1 and r_2 such that

$$0 < r_1 < \inf\{z : z \in \sigma(a)\}, \qquad r_2 > \sup\{z : z \in \sigma(a)\}$$

and

$$\frac{1}{r_1} > \sup\{z : z \in \sigma(a)\}, \qquad 0 < \frac{1}{r_2} < \inf\{z : z \in \sigma(a)\}.$$

Let γ be a closed rectifiable curve in $\{\mathrm{Re}\ z > 0\}$, which passes r_1 and r_2 and such that $\sigma(a) \subset \mathrm{ins}\ \gamma$. Then the curve $\frac{1}{\gamma} = \{\frac{1}{z} : z \in \gamma\}$ is also a closed rectifiable with $\sigma(a) \subset \mathrm{ins}\ \frac{1}{\gamma}$ and $\frac{1}{\gamma} \subset \{\mathrm{Re}\ z > 0\}$. Thus,

$$\begin{aligned}
(a^{-1})^\alpha &= \frac{1}{2\pi i}\int_\gamma z^\alpha (z-a^{-1})^{-1}dz \\
&= \frac{1}{2\pi i}\int_\gamma z^\alpha \left(a - \frac{1}{z}\right)^{-1}\frac{a}{z}dz \\
&= \frac{a}{2\pi i}\int_{\frac{1}{\gamma}} \lambda^{-\alpha-1}(\lambda-a)^{-1}d\lambda && \left(\text{Substituting} : \lambda = \frac{1}{z}\right) \\
&= aa^{-\alpha-1} = a^{-\alpha}. && \text{(Lemma 2)}
\end{aligned}$$

□

Lemma 4 *If* $0 < a \in A,\ 0 < b \in A, \quad \alpha, \beta \in \mathbb{R}$, *and* $ab = ba$, *then* $a^\alpha b^\beta = b^\beta a^\alpha$.

Proof Suppose that $z \notin \sigma(a)$, then $ab = ba \Longrightarrow (z-a)b = b(z-a) \Longrightarrow b(z-a)^{-1} = (z-a)^{-1}b$. Let γ be defined as in the discussion preceding lemma 1. Then

$$a^\alpha b = \left(\frac{1}{2\pi i}\int_\gamma z^\alpha (z-a)^{-1}dz\right) b = \frac{1}{2\pi i}\int_\gamma z^\alpha (z-a)^{-1} b dz$$

$$= \frac{1}{2\pi i}\int_\gamma z^\alpha b (z-a)^{-1}dz = b\left(\frac{1}{2\pi i}\int_\gamma z^\alpha (z-a)^{-1}dz\right) = ba^\alpha.$$

Thus,

$$ab = ba \implies a^\alpha b = ba^\alpha \implies a^\alpha b^\beta = b^\beta a^\alpha.$$

□

3. The arithmetic mean, geometric mean and harmonic mean

Naturally, for $a,\ b \in A$, and w_1, w_2 are positive numbers summing to 1, their weighted arithmetic mean can be defined as

$$A_w(a,b) := w_1 a + w_2 b.$$

If $a > 0,\ b > 0$, their weighted harmonic mean can be defined as

$$H_w(a,b) := \left(w_1 a^{-1} + w_2 b^{-1}\right)^{-1}.$$

From the point view of matrix analysis (see [1]), if $a > 0,\ b > 0$, and $w_1,\ w_2$ are positive numbers summing to 1, their weighted geometric mean can be defined as

$$G_w(a,b) := b^{\frac{1}{2}}(b^{-\frac{1}{2}}ab^{-\frac{1}{2}})^{w_1}b^{\frac{1}{2}}.$$

Denote $A_w(a,b),\ G_w(a,b)$ and $H_w(a,b)$ by $A(a,b),\ G(a,b)$ and $H(a,b)$ respectively if $w_1 = w_2 = \frac{1}{2}$. It is clear that $A_w(a,b),\ G_w(a,b),\ H_w(a,b) \in A$ and $H_w(a,b) > 0$ and $G_w(a,b) > 0$ by inequalities (2), (4), (5) and Lemma 1 above. Does the following arithmetic-geometric-harmonic inequalities hold

$$H_w(a,b) \le G_w(a,b) \le A_w(a,b)$$

in Banach $*$-algebras?

Based on the lemmas above we can prove some properties of arithmetic mean, geometric mean and harmonic mean mentioned by Ando [1].

Theorem 1 *Suppose that* $a,\ b \in A$ *with* $a > 0,\ b > 0$, *then*

$$H(a,b) = H(b,a) \qquad \text{and} \qquad G(a,b) = G(b,a).$$

Proof $H(a,b) = H(b,a)$ follows the definition of the harmonic mean and the fact that A is an Abelian group.

Observe that $G(a,b) = G(b,a)$ is equivalent to

$$a^{-\frac{1}{2}} b^{\frac{1}{2}} \left(b^{-\frac{1}{2}} a b^{-\frac{1}{2}}\right)^{\frac{1}{2}} b^{\frac{1}{2}} a^{-\frac{1}{2}} = \left(a^{-\frac{1}{2}} b a^{-\frac{1}{2}}\right)^{\frac{1}{2}}.$$

Since positive elements are equal if and only if their squares are equal (see [7], Lemma 6), using Lemma 2 this is in turn equivalent to

$$a^{-\frac{1}{2}} b^{\frac{1}{2}} \left(b^{-\frac{1}{2}} a b^{-\frac{1}{2}}\right)^{\frac{1}{2}} \left[b^{\frac{1}{2}} a^{-1} b^{\frac{1}{2}}\right] \left(b^{-\frac{1}{2}} a b^{-\frac{1}{2}}\right)^{\frac{1}{2}} b^{\frac{1}{2}} a^{-\frac{1}{2}} = a^{-\frac{1}{2}} b a^{-\frac{1}{2}}.$$

Since the term in square brackets is just $\left(b^{-\frac{1}{2}} a b^{-\frac{1}{2}}\right)^{-1}$ by Lemma 3, the left hand side of the expression above does indeed reduce to the right hand side when we use Lemma 2 again.

□

Theorem 2 *Suppose that* $a,\ b,\ c \in A$ *with* $a > 0,\ b > 0$ *and* $c \in \mathrm{Inv}(A)$, *then*

$$c^* H(a,b) c = H(c^*ac,\ c^*bc) \qquad \text{and} \qquad c^* G(a,b) c = G(c^*ac,\ c^*bc).$$

Proof Since $c \in \mathrm{Inv(A)}$, c^{-1} exists. Hence

$$c^* H(a,b) c = c^* \left(\frac{1}{2} a^{-1} + \frac{1}{2} b^{-1}\right)^{-1} c = \left(c^{-1} \left(\frac{1}{2} a^{-1} + \frac{1}{2} b^{-1}\right) (c^*)^{-1}\right)^{-1}$$

$$= \left(\frac{1}{2} c^{-1} a^{-1} (c^*)^{-1} + \frac{1}{2} c^{-1} b^{-1} (c^*)^{-1}\right)^{-1} = \left(\frac{1}{2} (c^*ac)^{-1} + \frac{1}{2} (c^*bc)^{-1}\right)^{-1} = H(c^*ac, c^*bc).$$

It is Analogous with the proof of Theorem 1, we now verify the second equality.

$$\begin{aligned}
& c^* G(a,b) c = G(c^*ac,\ c^*bc) \\
\Longleftrightarrow\ & c^* b^{\frac{1}{2}} \left(b^{-\frac{1}{2}} a b^{-\frac{1}{2}}\right)^{\frac{1}{2}} b^{\frac{1}{2}} c = (c^*bc)^{\frac{1}{2}} \left((c^*bc)^{-\frac{1}{2}} (c^*ac) (c^*bc)^{-\frac{1}{2}}\right)^{\frac{1}{2}} (c^*bc)^{\frac{1}{2}} \\
\Longleftrightarrow\ & (c^*bc)^{-\frac{1}{2}} c^* b^{\frac{1}{2}} \left(b^{-\frac{1}{2}} a b^{-\frac{1}{2}}\right)^{\frac{1}{2}} b^{\frac{1}{2}} c\, (c^*bc)^{-\frac{1}{2}} = \left((c^*bc)^{-\frac{1}{2}} (c^*ac) (c^*bc)^{-\frac{1}{2}}\right)^{\frac{1}{2}} \\
\Longleftrightarrow\ & \left((c^*bc)^{-\frac{1}{2}} c^* b^{\frac{1}{2}} \left(b^{-\frac{1}{2}} a b^{-\frac{1}{2}}\right)^{\frac{1}{2}} b^{\frac{1}{2}} c\, (c^*bc)^{-\frac{1}{2}}\right)^2 = (c^*bc)^{-\frac{1}{2}} (c^*ac) (c^*bc)^{-\frac{1}{2}}.
\end{aligned}$$

The last equality is true, since by Lemma 2

$$
\begin{aligned}
&\left((c^*bc)^{-\frac{1}{2}}\, c^* b^{\frac{1}{2}} \left(b^{-\frac{1}{2}} a b^{-\frac{1}{2}} \right)^{\frac{1}{2}} b^{\frac{1}{2}} c\, (c^*bc)^{-\frac{1}{2}} \right)^2 \\
= &\left((c^*bc)^{-\frac{1}{2}}\, c^* b^{\frac{1}{2}} \left(b^{-\frac{1}{2}} a b^{-\frac{1}{2}} \right)^{\frac{1}{2}} b^{\frac{1}{2}} c\, (c^*bc)^{-\frac{1}{2}} \right) \left((c^*bc)^{-\frac{1}{2}}\, c^* b^{\frac{1}{2}} \left(b^{-\frac{1}{2}} a b^{-\frac{1}{2}} \right)^{\frac{1}{2}} b^{\frac{1}{2}} c\, (c^*bc)^{-\frac{1}{2}} \right) \\
= &(c^*bc)^{-\frac{1}{2}}\, c^* b^{\frac{1}{2}} \left(b^{-\frac{1}{2}} a b^{-\frac{1}{2}} \right)^{\frac{1}{2}} b^{\frac{1}{2}} c\, (c^*bc)^{-1}\, c^* b^{\frac{1}{2}} \left(b^{-\frac{1}{2}} a b^{-\frac{1}{2}} \right)^{\frac{1}{2}} b^{\frac{1}{2}} c\, (c^*bc)^{-\frac{1}{2}} \\
= &(c^*bc)^{-\frac{1}{2}}\, c^* b^{\frac{1}{2}} \left(b^{-\frac{1}{2}} a b^{-\frac{1}{2}} \right)^{\frac{1}{2}} \left(b^{-\frac{1}{2}} a b^{-\frac{1}{2}} \right)^{\frac{1}{2}} b^{\frac{1}{2}} c\, (c^*bc)^{-\frac{1}{2}} \\
= &(c^*bc)^{-\frac{1}{2}}\, c^* b^{\frac{1}{2}} \left(b^{-\frac{1}{2}} a b^{-\frac{1}{2}} \right) b^{\frac{1}{2}} c\, (c^*bc)^{-\frac{1}{2}} \\
= &(c^*bc)^{-\frac{1}{2}}\, c^* a c\, (c^*bc)^{-\frac{1}{2}}\, .
\end{aligned}
$$

□

Theorem 3 *Suppose that* $a,\ b \in A$ *with* $a > 0,\ b > 0.$ *Then*

$$H_w(a,b)^{-1} = A_w(a^{-1}, b^{-1}) \qquad \text{and} \qquad G_w(a^{-1}, b^{-1}) = G_w(a,b)^{-1}.$$

Proof The first equality is obvious from its definitions. Using Lemma 2 and Lemma 3, we have

$$
\begin{aligned}
G_w(a^{-1}, b^{-1}) &= \left(b^{-1}\right)^{\frac{1}{2}} \left(\left(b^{-1}\right)^{-\frac{1}{2}} a^{-1} \left(b^{-1}\right)^{-\frac{1}{2}} \right)^{w_1} \left(b^{-1}\right)^{\frac{1}{2}} \\
&= \left(b^{\frac{1}{2}} \right)^{-1} \left(\left(b^{-\frac{1}{2}} a b^{-\frac{1}{2}} \right)^{-1} \right)^{w_1} \left(b^{\frac{1}{2}} \right)^{-1} \\
&= \left(b^{\frac{1}{2}} \left(b^{-\frac{1}{2}} a b^{-\frac{1}{2}} \right)^{w_1} b^{\frac{1}{2}} \right)^{-1} \\
&= G_w(a,b)^{-1}.
\end{aligned}
$$

□

Theorem 4 *Suppose that* $a,\ b \in A$ *with* $a > 0,\ b > 0,$ *and* w_1, w_2 *are positive numbers summing to* 1, *then*

$$H_w(a,b) \le G_w(a,b) \le A_w(a,b).$$

Proof Firstly we verify the arithmetic-geometric means inequality: $G_w(a,b) \le A_w(a,b)$. With the help of inequality (5),

$$\begin{aligned}
& G_w(a,b) \le A_w(a,b) \\
\Longleftrightarrow\ & b^{\frac{1}{2}}(b^{-\frac{1}{2}}ab^{-\frac{1}{2}})^{w_1}b^{\frac{1}{2}} \le w_1 a + w_2 b \\
\Longleftrightarrow\ & b^{\frac{1}{2}}(b^{-\frac{1}{2}}ab^{-\frac{1}{2}})^{w_1}b^{\frac{1}{2}} \le b^{\frac{1}{2}}\left(w_1 b^{-\frac{1}{2}}ab^{-\frac{1}{2}} + w_2\right)b^{\frac{1}{2}} \\
\Longleftrightarrow\ & (b^{-\frac{1}{2}}ab^{-\frac{1}{2}})^{w_1} \le w_1 b^{-\frac{1}{2}}ab^{-\frac{1}{2}} + w_2 \\
\Longleftrightarrow\ & w_1 n + w_2 - n^{w_1} \ge 0,
\end{aligned}$$

where $n := b^{-\frac{1}{2}}ab^{-\frac{1}{2}}$. Lemma 1 and inequality (5) imply $n > 0$, and hence $\sigma(n) \subset (0,\infty)$.

Let $f(z) = w_1 z + w_2 - z^{w_1}$, where z^{w_1} is the principal of the power function. Then $f(z)$ is analytic in the right half open plane $\{\mathrm{Re}\ z > 0\}$ of the complex plane. Next we claim that $f(z) \ge 0$ on the positive real line. In fact, let $x = z - 1$ in the Bernoulli inequality:

$$(1+x)^{w_1} \le 1 + w_1 x, \qquad \text{if } \ 0 < w_1 < 1 \ \text{ and } \ -1 < x.$$

We have

$$z^{w_1} \le w_1 z + (1 - w_1), \qquad \text{if } \ 0 < w_1 < 1 \ \text{ and } \ 0 < z,$$

that is,

$$f(z) \ge 0, \qquad \text{if } \ 0 < w_1 < 1 \ \text{ and } \ 0 < z.$$

The Spectral Mapping Theorem implies

$$\sigma(f(n)) = f(\sigma(n)) \subset [0,\infty).$$

So

$$f(n) = w_1 n + w_2 - n^{w_1} \ge 0.$$

Hence

$$G_w(a,b) \le A_w(a,b).$$

Replacing a and b by a^{-1} and b^{-1} respectively in the arithmetic-geometric means inequality, Theorem 3 and inequality (7) guarantees that

$$H_w(a,b) \le G_w(a,b).$$

□

In general, for $a_1, a_2, \ldots, a_n \in A$, and an n-tuple of positive numbers $w_1, w_2, \ldots, w_n$ are summing to 1, their weighted arithmetic mean in A can be defined as

$$A_w(a_1, a_2, \ldots, a_n) := w_1 a_1 + w_2 a_2 + \cdots + w_n a_n.$$

If $a_i > 0,\ 1 \le i \le n$, their weighted harmonic mean in A can be defined as

$$H_w(a_1, a_2, \ldots, a_n) := \left(w_1 a_1^{-1} + w_2 a_2^{-1} + \cdots + w_n a_n^{-1}\right)^{-1}.$$

From the point of view of matrix analysis (see [8]), if $a_i > 0,\ 1 \le i \le n$, and $w_1, \dots, w_n$ are positive numbers summing to 1, their weighted geometric mean in A can be defined as

$$G_w(a_1, a_2, \dots, a_n) := a_n^{\frac{1}{2}} (a_n^{-\frac{1}{2}} a_{n-1}^{\frac{1}{2}} \cdots (a_3^{-\frac{1}{2}} a_2^{\frac{1}{2}} (a_2^{-\frac{1}{2}} a_1 a_2^{-\frac{1}{2}})^{\alpha_1} a_2^{\frac{1}{2}} a_3^{-\frac{1}{2}})^{\alpha_2} \cdots a_{n-1}^{\frac{1}{2}} a_n^{-\frac{1}{2}})^{\alpha_{n-1}} a_n^{\frac{1}{2}},$$

where $\alpha_i = 1 - \left(w_{i+1} \ / \ \sum_{j=1}^{i+1} w_j \right)$ for $i = 1, \dots, n-1$. Note that this geometric mean is just the inductive generalization of $n = 2$ case, which was discussed in Theorem 3 and 4.

Based on Theorem 4 with the same inductive proof in [8], we have

Theorem 5 *Suppose that* $a_i \in A,\ 1 \le i \le n$, *with* $a_i > 0,\ 1 \le i \le n$, *and* $w_1, \dots, w_n$ *are positive numbers summing to* 1, *then*

$$H_w(a_1, \dots, a_n) \le G_w(a_1, \dots, a_n) \le A_w(a_1, \dots, a_n).$$

Acknowledgements Many thanks to the editor S. Stratila for his help.

References

[1] T. Ando, Concavity of certain maps on positive definite matrices and applications to Hadamard products. Linear Algebra Appl., **26**(1979) 203-241. MR **80f**: 15023.

[2] R. Bhatia, *Matrix analysis*, Springer–Verlag New York, 1997, MR **98i**: 15003.

[3] F. F. Bonsall and J. Duncan, Complete normed algebra. Spring-Verlag, 1973. MR **54** 11013.

[4] J. B. Conway, A course in functional analysis, second edition. Springer-Verlag New York, Inc. 1990. MR **91e**: 46001.

[5] T. Okayasu, The Löwner-Heinz inequality in Banach ∗-algebra. Glasgow Math. J. **42** (2000), 243-246. MR **2001a**: 46054.

[6] S. Shirali and J. W. M. Ford, Symmetry in complex involutory Banach algebras II. Duke Math. J. **37** (1970), 275-280. MR **41** 5977.

[7] K. Tanahashi and A. Uchiyama, The Furuta inequality in Banach ∗-algebras. Proc. Amer. Math. Soc., **128** (2000), 1691-1695. MR **2000j**: 47031.

[8] M. Sagae and K. Tanabe, Upper and lower bounds for the arithmetic-geometric-harmonic means of positive definite matrices, *Linear and Multilinear Algebra* **37**(1994) 279–282, MR **95m**: 15030.

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